Simple random walks on the integers conditioned to stay nonnegative

Vivien Despax

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Abstract

This note gathers some known results on one-dimensional simple random walks conditioned to stay nonnegative. We especially focus on two different notions of such conditionings in the zero drift case. This is the simplest case of the problem we consider in [D] in a multidimensional setting where we solve it by using tools from representation theory. It is worthy of interest to expose apart this particular one-dimensional case in details since it can be studied by using only elementary arguments.

Contents

1 Introduction

In the positive drift case, where the probability that a given simple random walk on the integers always stays nonnegative is positive, it is quite easy to define a Markov chain that can be naturally regarded as the corresponding random walk conditioned to always stay nonnegative : it only requires basic arguments from probability theory,

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mainly the strong law of large numbers. It is then not difficult to make explicit its law (see Theorem 1.3). One can then let the drift tends to zero in the transition probabilities obtained and so defines a first notion of conditioning for the centered random walk to always stay nonnegative. Now, if one starts directly with the centered simple random walk, the probability that it always stays nonnegative is zero. Nevertheless, we show that its transition probabilities under the conditioning that it stays nonnegative until an instant n have a limit when we let n tends to infinity. This is done by using a classical argument, namely the reflection principle, to derive an estimate on the tail distribution of the exit time from the nonnegative integers, see Theorem 2.6. This yields a second notion of conditioning for the centered random walk to always stay nonnegative and, at the end, one can observe that it coincides with the first one.

Basics on simple random walks (paragraph 1.1) and a first study of the positive drift case (paragraph 1.2) are included in the present introduction. In Section 2, we introduce the finite-time conditioning which is our main topic. After showing how the reflection principle can be used to recover the first result (paragraph 2.2.1), we use it to prove the convergence of the transition probabilities in the zero drift case (paragraph 2.2.2).

1.1 Simple random walks on the integers

Denote by P the lattice $\mathbb{Z}$. In the following, we implicitly endow the denumerable sets with the σ-algebra of all subsets. Let ν_0 be a probability measure on P with full support

$$\nu_0(\gamma) > 0 \qquad \gamma \in P.$$

Let p_{-1}, p_{+1} be two reals in $]0,1[$ such that $p_{-1} + p_{+1} = 1$, set $\theta = p_{-1}/p_{+1}$ and denote by $\nu = \nu(\theta)$ the probability measure

$$\begin{aligned}\nu &= p_{-1}\delta_{\{-1\}} + p_{+1}\delta_{\{+1\}} \\ &= \frac{\theta}{1+\theta}\delta_{\{-1\}} + \frac{1}{1+\theta}\delta_{\{+1\}}.\end{aligned}$$

Consider $(\mathcal{S}(0), \mathcal{X}(1), \mathcal{X}(2), \ldots)$ a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{T}, \mathbb{P})$ such that

1. $\mathcal{S}(0) : \Omega \to P$ has law ν_0. For any γ in P, we write $\mathbb{P}_\gamma$ instead of $\mathbb{P}_{\mathcal{S}(0)=\gamma}$ or $\mathbb{P}[\,.\,|\,\mathcal{S}(0) = \gamma]$ for the conditional probability defined by

$$\mathbb{P}_\gamma[\,.\,] = \frac{\mathbb{P}[\,.\cap \mathcal{S}(0) = \gamma]}{\mathbb{P}[\mathcal{S}(0) = \gamma]}.$$

2. $\mathcal{X} = (\mathcal{X}(n) : \Omega \to P)_{n\geq 1}$ are identically distributed with law ν. We denote by

$$\mathbf{d} = \mathbf{d}(\theta) = \mathbb{E}[\mathcal{X}(1)] = p_{+1} - p_{-1} = \frac{1-\theta}{1+\theta}$$

the common expectation of the random variables $\mathcal{X}$.

We define a sequence of random variables $\mathcal{S} = (\mathcal{S}(n))_{n\geq 1}$ by setting

$$\mathcal{S}(n) = \sum_{i=1}^{n} \mathcal{X}(i) \qquad n \geq 1.$$

The sequence of random variables $\mathcal{W} = (\mathcal{W}(n))_{n\geq 0}$ defined by $\mathcal{W}(0) = \mathcal{S}(0)$ and

$$\mathcal{W}(n) = \mathcal{S}(0) + \mathcal{S}(n) \qquad n \geq 1$$

is then called the simple (or nearest neighbors) random walk on P with initial distribution ν_0 and with law of increments ν. Its drift is $\mathbf{d}$. When it is zero (equivalently : when $\theta = 1$ or when $p_{-1} = p_{+1} = 1/2$), we obtain the centered (or symmetric) simple random walk on P with initial distribution ν_0. The sequence $\mathcal{W}$ is a (time-homogeneous) Markov chain with initial distribution ν_0, state space P and transition matrix $p = p(\theta)$ given by

$$\begin{aligned} p(\gamma,\Gamma) &= \mathbb{P}[\mathcal{W}(i+1) = \Gamma \,|\, \mathcal{W}(i) = \gamma] \\ &= \nu(\Gamma - \gamma) \\ &= \begin{cases} p_{+1} = \frac{1}{1+\theta} & \text{if } \Gamma - \gamma = +1 \\ p_{-1} = \frac{\theta}{1+\theta} & \text{if } \Gamma - \gamma = -1 \\ 0 & \text{otherwise} \end{cases} \qquad \gamma, \Gamma \in P,\ i \geq 0. \end{aligned}$$

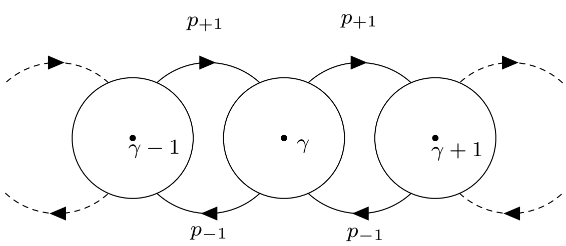

Figure 1: Graph of the Markov chain $\mathcal{W}$

Let P' be a denumerable set and p' a stochastic matrix on P', that is a map $p' : P' \times P' \to [0,1]$ such that $\sum_{y\in P'} p'(x,y) = 1$ for any x in P'. One can build a measurable space $(\Omega', \mathcal{T}')$ such that, for each probability measure ν_0' on $(\Omega', \mathcal{T}')$, there exists a probability measure $\mathbb{P}_{\nu_0'}$ on $(\Omega', \mathcal{T}')$ and a sequence of P'-valued maps defined on this probability space which is a Markov chain with initial distribution ν_0', state space P' and transition matrix p'. For a detailed construction, see [W] for example. We adopt this point of view to define a Markov chain from $\mathcal{W}$ that we call "the" random walk conditioned to stay nonnegative.

1.2 Conditioning to stay nonnegative : the positive drift case

Denote by C the open interval $]0,+\infty[$ and by $\overline{C}=[0,+\infty[$ its closure in $\mathbb{R}$. We also introduce

$$P^{+}=P\cap\overline{C}=\mathbb{Z}_{\geq 0}$$
$$P^{++}=P\cap C=\mathbb{Z}_{>0}.$$

Let us consider the event

$$\mathcal{W}\in\overline{C}=\bigcap_{n\geq 0}\mathcal{W}(n)\in\overline{C}=\bigcap_{n\geq 0}\mathcal{W}(n)\in P^{+}.$$

For any λ in P^{+}, we also denote by $\lambda+\mathcal{S}\in\overline{C}$ the event

$$\lambda+\mathcal{S}\in\overline{C}=\bigcap_{n\geq 1}\lambda+\mathcal{S}(n)\in\overline{C}=\bigcap_{n\geq 1}\lambda+\mathcal{S}(n)\in P^{+}.$$

The set $\overline{C}$ being stable under addition, notice that the sequence $\left(\lambda+\mathcal{S}\in\overline{C}\right)_{\lambda\in P^{+}}$ is increasing. Using independence and the previous remark, we have thus

$$\mathbb{P}_{\lambda}\left[\mathcal{W}\in\overline{C}\right]=\mathbb{P}\left[\lambda+\mathcal{S}\in\overline{C}\right]\geq\mathbb{P}\left[\mathcal{S}\in\overline{C}\right]=\mathbb{P}_{0}\left[\mathcal{W}\in\overline{C}\right]\qquad\lambda\in P^{+}\tag{1}$$

and the total probability formula gives then

$$\mathbb{P}\left[\mathcal{W}\in\overline{C}\right]=\sum_{\lambda\in P^{+}}\nu_{0}(\lambda)\,\mathbb{P}\left[\lambda+\mathcal{S}\in\overline{C}\right].\tag{2}$$

If the event $\mathcal{S}\in\overline{C}$ has a positive probability (as a by-product, Theorem 1.3 will give a necessary and sufficient condition on the drift), so has the event $\mathcal{W}\in\overline{C}$ by (1) and (2). Then, from a probabilistic point of view, to take the positive probability event $\mathcal{W}\in\overline{C}$ into account leads to introduce the conditional probability $\mathbb{P}_{\mathcal{W}\in\overline{C}}$ defined by

$$\mathbb{P}_{\mathcal{W}\in\overline{C}}\left[\,.\,\right]=\frac{\mathbb{P}\left[\,.\cap\mathcal{W}\in\overline{C}\right]}{\mathbb{P}\left[\mathcal{W}\in\overline{C}\right]}.$$

Let us observe how the transition probability between two states chosen in P^{+} is modified if one replaces $\mathbb{P}$ by $\mathbb{P}_{\mathcal{W}\in\overline{C}}$. One can easily derive the following proposition from (5) with Proposition 2.8 — its finite-time equivalent — or prove it directly.

Proposition 1.1 *Assume the event $\mathcal{S}\in\overline{C}$ has a positive probability. We have*

$$\mathbb{P}_{\mathcal{W}\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]=p(\lambda,\Lambda)\,\frac{\mathbb{P}_{\Lambda}\left[\mathcal{W}\in\overline{C}\right]}{\mathbb{P}_{\lambda}\left[\mathcal{W}\in\overline{C}\right]}\qquad\lambda,\Lambda\in P^{+},\ i\geq 0.$$

This proposition leads us to introduce the function

$$\begin{array}{cccc} h & : & P^{+} & \longrightarrow & \mathbb{R} \\ & & \lambda & \longmapsto & \mathbb{P}_{\lambda}\left[\mathcal{W}\in\overline{C}\right]=\mathbb{P}\left[\lambda+\mathcal{S}\in\overline{C}\right] \end{array}.$$

Observe that the transition probabilities of our random walk under $\mathbb{P}_{\mathcal{W}\in\overline{C}}$ are obtained by a particular transformation of the restriction of the matrix p to the subset P^+. In the general setting, given a pair (P', p') where P' is a denumerable set and p' is a substochastic matrix on P', that is a map $p' : P' \times P' \to [0,1]$ such that $\sum_{y\in P'} p'(x,y) \leq 1$ for any x in P', and a positive function $h' : P' \to \mathbb{R}_{>0}$, we can define another matrix $p'_{h'}$ on P' by setting

$$p'_{h'}(x,y) = p'(x,y)\frac{h'(y)}{h'(x)} \qquad x, y \in P'.$$

Then the matrix $p'_{h'}$ is stochastic on P' if and only if h' satisfies

$$h'(x) = \sum_{y\in P'} p'(x,y)\, h'(y) \qquad x \in P'.$$

In this case, we say that h' is a positive harmonic function on P' for p'. As we will see in the following, in the positive drift case, the function h that we introduced previously is an example of such positive harmonic function on P^+ for the restriction of the matrix p to P^+. Then, Proposition 1.1 shows that the map $p^+ = p^+(\theta)$ defined on $P^+ \times P^+$ by

$$p^+(\lambda,\Lambda) = p(\lambda,\Lambda)\frac{h(\Lambda)}{h(\lambda)} \qquad \lambda, \Lambda \in P^+$$

is a stochastic matrix on P^+. Any Markov chain associated to the pair (P^+, p^+) is called "the" random walk with law ν conditioned to always stay nonnegative and we will denote it by $\mathcal{W}^+$. We prove now two properties of the function h that will be sufficient to explicit it.

Proposition 1.2 *1. For any drift* $\mathbf{d}$*, the function* h *is harmonic on the set* P^+ *for the restriction of the matrix* p *to* P^+*, that is we have*

$$h(\lambda) = \begin{cases} p_{+1}h(1) & \text{if } \lambda = 0 \\ p_{+1}h(\lambda+1) + p_{-1}h(\lambda-1) & \text{if } \lambda \in P^{++} \end{cases}.$$

2. Assume $\mathbf{d}$ *is positive (equivalently : the parameter* θ *is in* $]0,1[$ *or* $p_{+1} > p_{-1}$*). We have*

$$h(\lambda) \xrightarrow[\lambda\to\infty]{} 1. \tag{3}$$

Proof.

1. We write

$$\begin{aligned} h(\lambda) &= \mathbb{P}\left[\lambda + \mathcal{S} \in \overline{C}\right] \\ &= \mathbb{P}\left[\mathcal{X}(1) = +1\right]\mathbb{P}_{\mathcal{X}(1)=+1}\left[\lambda + \mathcal{S} \in \overline{C}\right] + \mathbb{P}\left[\mathcal{X}(1) = -1\right]\mathbb{P}_{\mathcal{X}(1)=-1}\left[\lambda + \mathcal{S} \in \overline{C}\right] \\ &= p_{+1}\mathbb{P}_{\mathcal{X}(1)=+1}\left[\bigcap_{n\geq 2} \lambda + 1 + \sum_{i=2}^{n} \mathcal{X}(i) \in \overline{C}\right] \\ &+ p_{-1}\mathbb{P}_{\mathcal{X}(1)=-1}\left[\bigcap_{n\geq 2} \lambda - 1 + \sum_{i=2}^{n} \mathcal{X}(i) \in \overline{C}\right]. \end{aligned}$$

The random variables $\mathcal{X}$ being independent, we have

$$\mathbb{P}_{\mathcal{X}(1)=+1}\left[\bigcap_{n\geq 2}\lambda+1+\sum_{i=2}^{n}\mathcal{X}(i)\in\overline{C}\right]=\mathbb{P}\left[\bigcap_{n\geq 2}\lambda+1+\sum_{i=2}^{n}\mathcal{X}(i)\in\overline{C}\right]$$
$$\mathbb{P}_{\mathcal{X}(1)=-1}\left[\bigcap_{n\geq 2}\lambda-1+\sum_{i=2}^{n}\mathcal{X}(i)\in\overline{C}\right]=\mathbb{P}\left[\bigcap_{n\geq 2}\lambda-1+\sum_{i=2}^{n}\mathcal{X}(i)\in\overline{C}\right]$$

and, these variables being identically distributed, we have

$$\mathbb{P}\left[\bigcap_{n\geq 2}\lambda+1+\sum_{i=2}^{n}\mathcal{X}(i)\in\overline{C}\right]=\mathbb{P}\left[\bigcap_{n\geq 2}\lambda+1+\mathcal{S}(n-1)\in\overline{C}\right]=\mathbb{P}\left[\lambda+1+\mathcal{S}\in\overline{C}\right]$$
$$\mathbb{P}\left[\bigcap_{n\geq 2}\lambda-1+\sum_{i=2}^{n}\mathcal{X}(i)\in\overline{C}\right]=\mathbb{P}\left[\bigcap_{n\geq 2}\lambda-1+\mathcal{S}(n-1)\in\overline{C}\right]=\mathbb{P}\left[\lambda-1+\mathcal{S}\in\overline{C}\right].$$

The result follows.

2. According to the strong law of large numbers, we have

$$\frac{\mathcal{S}(n)}{n}\xrightarrow[n\to\infty]{}\mathbf{d}\qquad\mathbb{P}-a.s.$$

Since $\mathbf{d}$ is assumed to be positive, when the event $\mathcal{S}(n)/n\xrightarrow[n\to\infty]{}\mathbf{d}$ is fulfilled, there must exist a positive integer N such that $\mathcal{S}(n)$ is in $\overline{C}$ for any positive integer n with $n\geq N$. So

$$\frac{\mathcal{S}(n)}{n}\xrightarrow[n\to\infty]{}\mathbf{d}\subset\bigcup_{N\geq 1}\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}$$

from what we deduce

$$\mathbb{P}\left[\bigcup_{N\geq 1}\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}\right]=1.$$

The sequence of events $\left(\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}\right)_{N\geq 1}$ being increasing, this shows

$$\mathbb{P}\left[\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}\right]\xrightarrow[N\to\infty]{}1.\tag{4}$$

If N is a positive integer and if the event $\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}$ is fulfilled there must exist λ in P^{+} such that $\lambda+\mathcal{S}(n)$ is in $\overline{C}$ for any positive integer n. So

$$\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}\subset\bigcup_{\lambda\in P^{+}}\bigcap_{n\geq 1}\lambda+\mathcal{S}(n)\in\overline{C}=\bigcup_{\lambda\in P^{+}}\lambda+\mathcal{S}\in\overline{C}\qquad N\geq 1$$

and one has then

$$\mathbb{P}\left[\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}\right]\leq\mathbb{P}\left[\bigcup_{\lambda\in P^{+}}\lambda+\mathcal{S}\in\overline{C}\right]\qquad N\geq 1.$$

Reminding that the sequence of events $\left(\lambda+\mathcal{S}\in\overline{C}\right)_{\lambda\in P^{+}}$ is increasing, we deduce

$$\mathbb{P}\left[\bigcap_{n\geq N}\mathcal{S}(n)\in\overline{C}\right]\leq\lim_{\lambda\to\infty}\mathbb{P}\left[\lambda+\mathcal{S}\in\overline{C}\right]\qquad N\geq 1.$$

Then, the result follows from (4).

□

As said before, once these properties are established, the function h and, in the positive drift case, the matrix p^{+} can be easily computed explicitly.

Theorem 1.3 *1. Assume the drift* $\mathbf{d}$ *is positive (equivalently : the parameter* θ *is in* $]0,1[$ *or* $p_{+1}>p_{-1}$*). We have*

$$h(\lambda)=1-\theta^{\lambda+1}\qquad\lambda\in P^{+}.$$

In particular, we have $\mathbb{P}_{0}\left[\mathcal{W}\in\overline{C}\right]=\mathbb{P}\left[\mathcal{S}\in\overline{C}\right]=1-\theta>0$ *and the transition matrix* p^{+} *of* $\mathcal{W}^{+}$ *is given by*

$$p^{+}(\lambda,\Lambda)=p(\lambda,\Lambda)\frac{1-\theta^{\Lambda+1}}{1-\theta^{\lambda+1}}=\begin{cases}\frac{1}{1+\theta}\frac{1-\theta^{\lambda+2}}{1-\theta^{\lambda+1}} & \textit{if } \Lambda-\lambda=+1\\ \frac{\theta}{1+\theta}\frac{1-\theta^{\lambda}}{1-\theta^{\lambda+1}} & \textit{if } \Lambda-\lambda=-1\\ 0 & \textit{otherwise}\end{cases}\qquad\lambda,\Lambda\in P^{+}.$$

2. Assume the drift is zero (equivalently : $\theta=1$*). Then* $h=0$*. Thus we have* $\mathbb{P}_{\lambda}\left[\mathcal{W}\in\overline{C}\right]=0$ *for any* λ *in* P^{+} *and so* $\mathbb{P}\left[\mathcal{W}\in\overline{C}\right]=0$*.*

Proof. Of course, the first part of Proposition 1.2 tells us that the fonction h is a solution of a linear recurrence of order 2 with constant coefficients and we can solve it with standard methods. As we are dealing with probabilities, one can also prefer to use this proposition in order to write

$$h(\lambda)=\frac{1}{1+\theta}h(\lambda+1)+\frac{\theta}{1+\theta}h(\lambda-1)\qquad\lambda\in P^{++},$$

that is

$$\frac{1}{1+\theta}h(\lambda)+\frac{\theta}{1+\theta}h(\lambda)=\frac{1}{1+\theta}h(\lambda+1)+\frac{\theta}{1+\theta}h(\lambda-1)\qquad\lambda\in P^{++},$$

or

$$h(\lambda+1)-h(\lambda)=\theta\left(h(\lambda)-h(\lambda-1)\right)\qquad\lambda\in P^{++}.$$

We get then

$$h(\lambda+1)-h(\lambda)=\theta^{\lambda}(h(1)-h(0))=\theta^{\lambda+1}h(0)\qquad \lambda\in P^{++}.$$

Notice that the previous relation still holds when λ is 0.

1. If the parameter θ is in $]0,1[$, we deduce

$$\begin{aligned}h(\lambda)-h(0)&=\left(\sum_{\Lambda=0}^{\lambda-1}\theta^{\Lambda+1}\right)h(0)\\&=\theta\frac{1-\theta^{\lambda}}{1-\theta}h(0)\qquad \lambda\in P^{++}.\end{aligned}$$

   Once again, this relation still holds when λ is 0, so we have

$$h(\lambda)=h(0)\,\frac{1-\theta^{\lambda+1}}{1-\theta}\qquad \lambda\in P^{+}.$$

   the second part of Proposition 1.2 gives the expected formula for h. Using then Proposition 1.1, we obtain p^{+}.

2. If $\theta=1$, we must have $h(0)=0$ since h is bounded on P^{+}.

□

1.3 Conclusions and a question

When the drift is positive, observe that we have

$$\begin{cases}p^{+}(\lambda,\lambda+1)\xrightarrow[\lambda\to\infty]{}p(\lambda,\lambda+1)\\ p^{+}(\lambda,\lambda-1)\xrightarrow[\lambda\to\infty]{}p(\lambda,\lambda-1)\end{cases}.$$

This corresponds to the intuitive fact that, in this case, the influence of the conditioning to always stay nonnegative must disappear when one computes the transition probabilities between two states which are far away from the origin.

The previous construction of $\mathcal{W}^{+}$ is known as the Doob's h-transform approach. Remark that in the positive drift case, the proof of Theorem 1.3 shows that the set of positive harmonic functions on P^{+} for the restriction of p^{+} to P^{+} is reduced to the ch where c is a positive real. This may a posteriori justify our interest on the conditioning for the random walk to always stay nonnegative : when the drift is positive, this is the only one to consider in Doob's sense.

In the nonpositive drift case, Theorem 1.3 tells us that Doob's approach does not work anymore — or at least not so naively — since $h=0$. Nevertheless, we observe that the map p^+ pointwise converges on $P^+ \times P^+$ as the parameter θ tends to 1 from the left, that is when the drift $\mathbf{d}$ tends to zero from the right : the limit is given by

$$p^+(\theta)(\lambda,\Lambda) \xrightarrow[\theta\in]0,1[]{\theta\to 1} p(1)(\lambda,\Lambda)\frac{\Lambda+1}{\lambda+1} = \begin{cases} \frac{1}{2}\frac{\lambda+2}{\lambda+1} & \text{if } \Lambda=\lambda+1 \\ \frac{1}{2}\frac{\lambda}{\lambda+1} & \text{if } \Lambda=\lambda-1 \\ 0 & \text{otherwise} \end{cases} \qquad \lambda,\Lambda\in P^+.$$

(Once again, the influence of the conditioning disappears asymptotically.)

Can we modify this natural first approach in order to build a definition of a random walk conditioned to always stay nonnegative when $\mathbf{d}\geq 0$ that will both coincides with the previous one when $\mathbf{d}>0$ and with its limit when $\mathbf{d}=0$?

2 A finite-time approach

When the drift is zero, we know that the event $\mathcal{W}\in\overline{C}$ has probability zero. We introduce $(\mathcal{W}(\leq n))_{n\geq 0}$ the increasing sequence of positive probability events defined by

$$\mathcal{W}(\leq n)\in\overline{C} = \bigcap_{i=0}^{n}\mathcal{W}(i)\in\overline{C} = \bigcap_{i=1}^{n}\mathcal{W}(i)\in P^+ \qquad n\geq 0$$

which limit is $\mathcal{W}\in\overline{C}$. We also introduce for each λ in P^+ the increasing sequence $(\lambda+\mathcal{S}(\leq n))_{n\geq 1}$ defined by

$$\lambda+\mathcal{S}(\leq n)\in\overline{C} = \bigcap_{i=1}^{n}\lambda+\mathcal{S}(i)\in\overline{C} = \bigcap_{i=1}^{n}\lambda+\mathcal{S}(i)\in P^+ \qquad n\geq 1$$

which limit is $\lambda+\mathcal{S}\in\overline{C}$. Observe that we have

$$\mathbb{P}[\mathcal{W}(\leq n)] = \sum_{\lambda\in P^+}\nu_0(\lambda)\,\mathbb{P}_\lambda[\mathcal{W}(\leq n)] \qquad n\geq 0,$$

$$\mathbb{P}_\lambda[\mathcal{W}(\leq n)] = \mathbb{P}[\lambda+\mathcal{S}(\leq n)] \qquad \lambda\in P^+,\ n\geq 1,$$

$$\mathbb{P}\left[\lambda+\mathcal{S}(\leq n)\in\overline{C}\right] \geq \mathbb{P}\left[\mathcal{S}(\leq n)\in\overline{C}\right] \geq \mathbb{P}\left[\bigcap_{i=1}^{n}\mathcal{X}(i)=+1\right] = p_{+1}^{n} > 0 \qquad \lambda\in P^+,\ n\geq 1.$$

Then we can recover the result of Theorem 1.3 by writing

$$\mathbb{P}_\lambda\left[\mathcal{W}(\leq n)\in\overline{C}\right] \xrightarrow[n\to\infty]{} \mathbb{P}_\lambda\left[\mathcal{W}\in\overline{C}\right] = h(\lambda) \qquad \lambda\in P^+ \tag{5}$$

and by exploiting the reflection principle described above to rewrite the probabilities

$$\mathbb{P}_\lambda\left[\mathcal{W}(\leq n)\in\overline{C}\right] \qquad \lambda\in P^+,\ n\geq 1$$

in an advantageous manner. Moreover, this finite-time approach can also be exploited in the zero drift case and allows us to give a positive answer to the question formulated in paragraph 1.3.

Let us introduce the following sequence $(h_n)_{n\geq 1}$

$$\begin{array}{cccc} h_n & : & P^+ & \longrightarrow & \mathbb{R} \\ & & \lambda & \longmapsto & \mathbb{P}_\lambda\left[\mathcal{W}(\leq n)\in\overline{C}\right]=\mathbb{P}\left[\lambda+\mathcal{S}(\leq n)\in\overline{C}\right] \end{array} \qquad n\geq 1.$$

2.1 The reflection principle

Given γ,Γ in P and a positive integer n, we denote by $\pi_n(\gamma,\Gamma)$ the set

$$\left\{(\pi_0,\pi_1,\dots,\pi_n)\in P^{n+1}:(\pi_0,\pi_n)=(\gamma,\Gamma),\pi_{i+1}-\pi_i\in\{-1,+1\}\quad 0\leq i\leq n-1\right\},$$

that is the set of paths in P of length n with steps in $\{-1,+1\}$ starting at γ and ending at Γ. We will also intoduce $\pi_n^0(\gamma,\Gamma)$, the subset of paths hitting 0 at some instant, and $\pi_n^+(\lambda,\Lambda)$, the subset of paths always staying in P^+. Let γ,Γ be in P and consider a positive integer n. We make some simple observations about the set $\pi_n(\gamma,\Gamma)$. Actually, the second and the third facts are the main ingredients of the reflection principle. The reader is referred to [GZ] for a presentation of this argument in its original setting.

1. It is non-empty if and only if the three integers γ,Γ,n are such that

$$\begin{cases}\gamma-n\leq\Gamma\leq\gamma+n\\ \gamma+n=\Gamma\mod 2\end{cases}.$$

2. If a path visits both sets P^{++} and $-P^+=\mathbb{Z}_{\leq 0}$, then it must visit 0. Thus, when γ is in P^{++}, the set
$$\pi_n(\gamma,\Gamma)\setminus\pi_n^0(\gamma,\Gamma)$$
is the subset of paths remaining in P^{++}.

3. Suppose γ,Γ are in P^{++}. If $\pi_n^0(\gamma,\Gamma)$ is nonempty then we must have $n\geq 2$. To each $\pi=(\pi_0,\pi_1,\pi_2,\dots,\pi_n)$ in $\pi_n^0(\gamma,\Gamma)$ we can associate i_0, the greatest integer i in $\{0,\dots,n\}$ such that $\pi_i=0$. Actually, the integer i_0 is in $\{1,\dots,n-1\}$. Define then another path $\overline{\pi}=(\overline{\pi}_0,\overline{\pi}_1,\overline{\pi}_2,\dots,\overline{\pi}_n)$ by setting

$$\overline{\pi}_i=\begin{cases}-\pi_i & \text{if } 0\leq i\leq i_0-1\\ \pi_i & \text{if } i_0\leq i\leq n\end{cases}.$$

The path $\overline{\pi}$ is clearly in $\pi_n^0(-\gamma,\Gamma)$ and i_0 is the greatest integer i in $\{1,\dots,n-1\}$ such that $\overline{\pi}_i=0$. We have

$$(-\overline{\pi}_0,\dots,-\overline{\pi}_{i_0-1},\overline{\pi}_{i_0},\overline{\pi}_{i_0+1},\dots,\overline{\pi}_n)=\pi.$$

Thus, there exists a bijection between the two finite sets $\pi_n^0(\gamma,\Gamma)$ and $\pi_n^0(-\gamma,\Gamma)$. But $-\gamma$ being in $-P^{++}=\mathbb{Z}_{<0}$ and Γ being in P^{++}, we have

$$\pi_n^0(-\gamma,\Gamma)=\pi_n(-\gamma,\Gamma).$$

In particular, one has

$$\operatorname{Card}\pi_n^0(\gamma,\Gamma)=\operatorname{Card}\pi_n(-\gamma,\Gamma). \tag{6}$$

Remark that this equality still holds when $\pi_n^0(\gamma,\Gamma)$ is empty since $\pi_n(-\gamma,\Gamma)$ is also empty in this case.

Theorem 2.1 (André – Gessel, Zeilberger) *We have*

$$\pi_n^+(\lambda,\Lambda)=\pi_n(\lambda+1,\Lambda+1)-\pi_n(-\lambda-1,\Lambda+1) \qquad \lambda,\Lambda\in P^+,\ n\geq 1.$$

Remark 2.2 *As an application of this principle, one may deduce the formula*

$$\pi_n^+(\lambda,\Lambda)=\binom{n}{\frac{n+\Lambda-\lambda}{2}}-\binom{n}{\frac{n+\Lambda+\lambda+2}{2}} \qquad \lambda,\Lambda\in P^+,\ n\geq 1,\ \begin{cases}\lambda-n\leq\Lambda\leq\lambda+n\\ \lambda+n\equiv\Lambda \mod 2\end{cases}.$$

We will not need it in the following.

2.2 Application to the simple random walks

Corollary 2.3 *We have*

$$h_n(\lambda)=\mathbb{P}_{\lambda+1}[\mathcal{W}(n)\in C]-\theta^{\lambda+1}\mathbb{P}_{-\lambda-1}[\mathcal{W}(n)\in C] \qquad \lambda\in P^+,\ n\geq 1.$$

Proof. One has

$$\begin{aligned}\mathbb{P}\left[\lambda+\mathcal{S}(\leq n)\in\overline{C}\right]&=\sum_{\Lambda\in P^+}\mathbb{P}\left[\lambda+\mathcal{S}(\leq n)\in\overline{C}\cap\mathcal{S}(n)=\Lambda\right]\\&=\sum_{\Lambda\in P^+}\left(\sum_{\substack{\pi\in\pi_n^+(\lambda,\Lambda)\\ \pi=(\pi_0,\pi_1,\dots,\pi_n)}}\mathbb{P}\left[\bigcap_{i=1}^n\lambda+\mathcal{S}(i)=\pi_i\right]\right).\end{aligned}$$

Given Λ in P^+ and $\pi=(\pi_0,\pi_1,\dots,\pi_n)$ in $\pi_n^+(\lambda,\Lambda)$, one writes

$$\mathbb{P}\left[\bigcap_{i=1}^n\lambda+\mathcal{S}(i)=\pi_i\right]=p_{+1}^{\frac{n+(\Lambda-\lambda)}{2}}p_{-1}^{\frac{n-(\Lambda-\lambda)}{2}}.$$

As this probability does not depend on the choice of π in $\pi_n^+(\lambda,\Lambda)$, we have then

$$\sum_{\substack{\pi\in\pi_n^+(\lambda,\Lambda)\\ \pi=(\pi_0,\pi_1,\dots,\pi_n)}}\mathbb{P}\left[\bigcap_{i=1}^n\lambda+\mathcal{S}(i)=\pi_i\right]=\operatorname{Card}\pi_n^+(\lambda,\Lambda)\,p_{+1}^{\frac{n+(\Lambda-\lambda)}{2}}p_{-1}^{\frac{n-(\Lambda-\lambda)}{2}}.$$

But λ and Λ being in P^+, we can use Theorem 2.1 to write

$$\begin{aligned}
&\sum_{\substack{\pi\in\pi_n^+(\lambda,\Lambda)\\ \pi=(\pi_0,\pi_1,\dots,\pi_n)}} \mathbb{P}\left[\bigcap_{i=1}^{n}\lambda+\mathcal{S}(i)=\pi_i\right]\\
&=\operatorname{Card}\pi_n(\lambda+1,\Lambda+1)\,p_{+1}^{\frac{n+(\Lambda-\lambda)}{2}}p_{-1}^{\frac{n-(\Lambda-\lambda)}{2}}-\operatorname{Card}\pi_n(-\lambda-1,\Lambda+1)\,p_{+1}^{\frac{n+(\Lambda-\lambda)}{2}}p_{-1}^{\frac{n-(\Lambda-\lambda)}{2}}\\
&=\operatorname{Card}\pi_n(\lambda+1,\Lambda+1)\,p_{+1}^{\frac{n+(\Lambda+1-(\lambda+1))}{2}}p_{-1}^{\frac{n-(\Lambda+1-(\lambda+1))}{2}}\\
&-\theta^{\lambda+1}\operatorname{Card}\pi_n(-\lambda-1,\Lambda+1)\,p_{+1}^{\frac{n+(\Lambda+1-(-\lambda-1))}{2}}p_{-1}^{\frac{n-(\Lambda+1-(-\lambda-1))}{2}}\\
&=\mathbb{P}[\lambda+1+\mathcal{S}(n)=\Lambda+1]-\theta^{\lambda+1}\mathbb{P}[-\lambda-1+\mathcal{S}(n)=\Lambda+1].
\end{aligned}$$

Summing over Λ in P^+, we finally obtain the expected equality. □

2.2.1 The positive drift case

Lemma 2.4 *Assume the drift* $\mathbf{d}$ *is in* C *(equivalently : the parameter* θ *is in* $]0,1[$ *or* $p_{+1}>p_{-1}$*). We have*

$$\mathbb{P}_\gamma[\mathcal{W}(n)\in C]\xrightarrow[n\to\infty]{}1\qquad\gamma\in P.$$

Proof. According to the strong law of large numbers, once again, we have

$$\mathbb{P}\left[\frac{\mathcal{S}(n)}{n}\xrightarrow[n\to\infty]{}\mathbf{d}\right]=1.$$

When the event $\frac{\mathcal{S}(n)}{n}\xrightarrow[n\to\infty]{}\mathbf{d}$ is fulfilled, the drift $\mathbf{d}$ being positive, there exists a positive integer N such that $\gamma+\mathcal{S}(n)$ is in C for any positive integer n with $n\geq N$. Thus we have the inclusion

$$\frac{\mathcal{S}(n)}{n}\xrightarrow[n\to\infty]{}\mathbf{d}\subset\bigcup_{N\geq1}\bigcap_{n\geq N}\gamma+\mathcal{S}(n)\in C. \tag{7}$$

The sequence of events $\left(\bigcap_{n\geq N}\gamma+\mathcal{S}(n)\in C\right)_{N\geq1}$ being increasing, one has

$$\mathbb{P}\left[\bigcap_{n\geq N}\gamma+\mathcal{S}(n)\in C\right]\xrightarrow[N\to\infty]{}\mathbb{P}\left[\bigcup_{N\geq1}\bigcap_{n\geq N}\gamma+\mathcal{S}(n)\in C\right].$$

Now inclusions (7) and

$$\bigcap_{n\geq N}\gamma+\mathcal{S}(n)\in C\subset\gamma+\mathcal{S}(N)\in C\qquad N\geq1$$

finally give

$$\mathbb{P}[\gamma+\mathcal{S}(N)\in C]\xrightarrow[N\to\infty]{}1.$$

□

Corollary 2.5 *Assume the drift* $\mathbf{d}$ *is in* C *(equivalently : the parameter* θ *is in* $]0,1[$ *or* $p_{+1} > p_{-1}$*). The sequence* $(h_n)_{n\geq 1}$ *pointwise converges on* P^+ *and its limit is given by*

$$h_n(\lambda) \xrightarrow[n\to\infty]{} 1-\theta^{\lambda+1} \qquad \lambda \in P^+.$$

We recover in particular the the first conclusion of Theorem 1.3.

Proof. It is a straightforward consequence of (5), Corollary 2.3 and Lemma 2.4. □

2.2.2 The zero drift case

Theorem 2.6 *Assume the drift is zero (equivalently :* $\theta = 1$ *or* $p_{+1} = p_{-1} = 1/2$*). We have*

$$h_n(\lambda) \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\,(\lambda+1)\,\frac{1}{\sqrt{n}} \qquad \lambda \in P^+.$$

Proof. As $\theta = 1$, Corollary 2.3 and Lemma 2.4 show that $(h_n)_{n\geq 1}$ pointwise converges to zero on P^+. We recover in particular the the second conclusion of Theorem 1.3 in the zero drift case. Nevertheless, one can write

$$\begin{aligned} h_n(\lambda) &= \mathbb{P}[\lambda+1+\mathcal{S}(n) \in C] - \mathbb{P}[-\lambda-1+\mathcal{S}(n) \in C] \\ &= \mathbb{P}[\mathcal{S}(n) \geq -\lambda] - \mathbb{P}[\mathcal{S}(n) \geq \lambda+2] \\ &= \sum_{i=-\lambda}^{\lambda+1} \mathbb{P}[\mathcal{S}(n) = i] \qquad \lambda \in P^+,\ n \geq 1. \end{aligned}$$

Let us fix a positive integer n and λ in P^+. Remind that the law of the random variable $\frac{\mathcal{S}(n)+n}{2}$ is the binomial ditribution with parameters n and $1/2$. Then one has

$$\mathbb{P}[\mathcal{S}(n) = i] = \begin{cases} \frac{1}{2^n}\binom{n}{\frac{n+i}{2}} & \text{if } -n \leq i \leq n \text{ and } i = n \mod 2 \\ 0 & \text{otherwise} \end{cases} \qquad -\lambda \leq i \leq \lambda+1.$$

Let us assume that $n \geq \lambda + 2$. We have

$$h_n(\lambda) = \sum_{\substack{-\lambda\leq i\leq\lambda+1 \\ -n\leq i\leq n \\ i=n \mod 2}} \frac{1}{2^n}\binom{n}{\frac{n+i}{2}} = \sum_{\substack{-\lambda\leq i\leq\lambda+1 \\ i=n \mod 2}} \frac{1}{2^n}\binom{n}{\frac{n+i}{2}}.$$

Let us fix an integer i such that $-\lambda \leq i \leq \lambda+1$ and $i = n \mod 2$. By using Stirling's approximation of factorials, we know that there exists a sequence of reals $(\epsilon_m)_{m\geq 1}$ such that

$$\begin{cases} m! = \sqrt{2\pi}\sqrt{m}\left(\frac{m}{\mathrm{e}}\right)^m (1+\epsilon_m) & m \geq 1 \\ \epsilon_m \xrightarrow[m\to\infty]{} 0 \end{cases}$$

Thus, one can write

$$\frac{1}{2^n}\binom{n}{\frac{n+i}{2}} = \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{n}}\left(\sqrt{\frac{1-\frac{i}{n}}{1+\frac{i}{n}}}\right)^i \frac{\exp\left(-\frac{n}{2}\ln\left(1-\left(\frac{i}{n}\right)^2\right)\right)}{\sqrt{1-\left(\frac{i}{n}\right)^2}} \frac{1+\epsilon_n}{\left(1+\epsilon_{\frac{n+i}{2}}\right)\left(1+\epsilon_{\frac{n-i}{2}}\right)}.$$

Hence, for any positive integer n large enough, we have

$$\frac{h_{2n}(\lambda)}{\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2n}}} = \left(\sum_{\substack{-\lambda\leq i\leq\lambda+1 \\ i \text{ even}}} \frac{\left(1+\frac{i}{2n}\right)^{\frac{-i-1}{2}}\left(1-\frac{i}{2n}\right)^{\frac{i-1}{2}}\exp\left(-n\ln\left(1-\frac{i^2}{4n^2}\right)\right)}{\left(1+\epsilon_{\frac{2n+i}{2}}\right)\left(1+\epsilon_{\frac{2n-i}{2}}\right)}\right)(1+\epsilon_{2n}) \quad (8)$$

and

$$\frac{h_{2n-1}(\lambda)}{\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2n-1}}}$$
$$= \left(\sum_{\substack{-\lambda\leq i\leq\lambda+1 \\ i \text{ odd}}} \frac{\left(1+\frac{i}{2n-1}\right)^{\frac{-i-1}{2}}\left(1-\frac{i}{2n-1}\right)^{\frac{i-1}{2}}\exp\left(\frac{1}{2}\ln\left(1-\frac{i^2}{(2n-1)^2}\right)\right)\exp\left(-n\ln\left(1-\frac{i^2}{(2n-1)^2}\right)\right)}{\left(1+\epsilon_{\frac{2n-1+i}{2}}\right)\left(1+\epsilon_{\frac{2n-1-i}{2}}\right)}\right)$$
$$\times(1+\epsilon_{2n-1}). \quad (9)$$

In both cases, there are exactly $\lambda+1$ terms in the sums (8) (9) and each of them converges to 1 when n goes to infinity. Thus, both subsequences

$$\left(\frac{h_{2n}(\lambda)}{\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2n}}}\right)_{n\geq 1} \qquad \left(\frac{h_{2n-1}(\lambda)}{\sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{2n-1}}}\right)_{n\geq 1}$$

converge to $\lambda+1$. □

Remark 2.7 *As we have*

$$h_n(\lambda) = \mathbb{P}[\tau_\lambda > n] \qquad \lambda\in P^+,\ n\geq 1$$

where

$$\tau_\lambda = \inf\left\{n\geq 1 : \lambda+\mathcal{S}(n)\notin\overline{C}\right\} \qquad \lambda\in P^+,$$

this result can be regarded as an estimate of the tail distribution of the exit time from the cone $\overline{C}$ for the centered simple random walk. Recently, Denisov and Wachtel gave a multidimensional version for more general centered random walks and more general cones. Their approach is completely different from ours, see [DW]. Actually, we use their result in our article [D].

When the drift is zero, we already know by (5) and Theorem 1.3 that each sequence $\left(\mathbb{P}_\lambda\left[\mathcal{W}(n)\in\overline{C}\right]\right)_{n\geq 0}$ must converge to zero, but now we have an idea of the speed of this convergence. We exploit it in the following.

As said at the begining of Section 2, we can use positive probability approximations of the event $\mathcal{W}\in\overline{C}$ in order to build a notion of conditioning of staying nonnegative in the zero drift case. Indeed, we can introduce the sequence of conditional probabilities $\left(\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\right)_{n\geq 0}$ defined by

$$\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\,.\,\right]=\frac{\mathbb{P}\left[\,.\cap\mathcal{W}(\leq n)\in\overline{C}\right]}{\mathbb{P}\left[\mathcal{W}(\leq n)\in\overline{C}\right]}\qquad n\geq 0.$$

Since the sequence $\left(\mathbb{P}\left[\mathcal{W}(\leq n)\in\overline{C}\right]\right)_{n\geq 1}$ converges to $\mathbb{P}\left[\mathcal{W}\in\overline{C}\right]$, one has, under the assumption that $\mathbb{P}\left[\mathcal{W}\in\overline{C}\right]$ is positive (that is when the drift is positive or when θ is in $]0,1[$),

$$\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[E\right]\xrightarrow[n\to\infty]{}\mathbb{P}_{\mathcal{W}\in\overline{C}}\left[E\right]\qquad E\in\mathcal{T}.$$

If we set

$$p_n^+(\lambda,\Lambda,i)=\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]\qquad \lambda,\Lambda\in P^+,\ n\geq 0,\ i\geq 0,$$

the previous remark gives in particular

$$p_n^+(\lambda,\Lambda,i)\xrightarrow[n\to\infty]{}p^+(\lambda,\Lambda)\qquad \lambda,\Lambda\in P^+,\ i\geq 0$$

whenever the parameter θ is in $]0,1[$, that is in the positive drift case. When the drift is zero, that is when the simple random walk considered is the centered one, it is no longer obvious whether the sequences

$$\left(\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]\right)_{n\geq 0}\qquad \lambda,\Lambda\in P^+,\ i\geq 0$$

are convergent, since the denominator $\mathbb{P}\left[\mathcal{W}(\leq n)\in\overline{C}\right]$ tends to zero when n tends to infinity. As done before in the positive drift case, we start by observing how the transition probability between two states in P^+ is modified when we replace $\mathbb{P}$ by one of the $\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}$.

Proposition 2.8 *Let λ,Λ be in P^+. One has*

$$p_n^+(\lambda,\Lambda,i)=\begin{cases}p(\lambda,\Lambda)\,\frac{h_{n-i-1}(\Lambda)}{h_{n-i}(\lambda)} & \text{if } i\leq n-2\\ p(\lambda,\Lambda) & \text{if } i>n-2\end{cases}\qquad n\geq 2,\ i\geq 0.$$

In particular, one has for any nonnegative integer i

$$p_n^+(\lambda,\Lambda,i)=p_{n-i}^+(\lambda,\Lambda,0)\qquad n\geq i+2.$$

Proof. One has

$$\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]=\frac{\mathbb{P}\left[\mathcal{W}(i+1)=\Lambda\cap\mathcal{W}(i)=\lambda\cap\mathcal{W}(\leq n)\in\overline{C}\right]}{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\mathcal{W}(\leq n)\in\overline{C}\right]}.$$

Assume first that $i>n-2$. The Markov property gives

$$\frac{\mathbb{P}\left[\mathcal{W}(i+1)=\Lambda\cap\mathcal{W}(i)=\lambda\cap\mathcal{W}(\leq n)\in\overline{C}\right]}{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\mathcal{W}(\leq n)\in\overline{C}\right]}=p(\lambda,\Lambda).$$

Assume now that $i\leq n-2$. One writes

$$\begin{aligned}
&\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]\\
&=\frac{\mathbb{P}\left[\mathcal{W}(\leq i-1)\in\overline{C}\cap\mathcal{W}(i+1)=\Lambda\cap\mathcal{W}(i)=\lambda\cap\left(\bigcap_{j=i+2}^{n}\mathcal{W}(j)\in\overline{C}\right)\right]}{\mathbb{P}\left[\mathcal{W}(\leq i-1)\in\overline{C}\cap\mathcal{W}(i)=\lambda\cap\left(\bigcap_{j=i+1}^{n}\mathcal{W}(j)\in\overline{C}\right)\right]}\\
&=\frac{\mathbb{P}\left[\mathcal{W}(i+1)=\Lambda\cap\left(\bigcap_{j=i+2}^{n}\mathcal{W}(j)\in\overline{C}\right)\,|\,\mathcal{W}(\leq i-1)\in\overline{C}\cap\mathcal{W}(i)=\lambda\right]}{\mathbb{P}\left[\bigcap_{j=i+1}^{n}\mathcal{W}(j)\in\overline{C}\,|\,\mathcal{W}(\leq i-1)\in\overline{C}\cap\mathcal{W}(i)=\lambda\right]}.
\end{aligned}$$

Using Markov property, one gets

$$\begin{aligned}
&\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]\\
&=\frac{\mathbb{P}\left[\mathcal{W}(i+1)=\Lambda\cap\left(\bigcap_{j=i+2}^{n}\mathcal{W}(j)\in\overline{C}\right)\,|\,\mathcal{W}(i)=\lambda\right]}{\mathbb{P}\left[\bigcap_{j=i+1}^{n}\mathcal{W}(j)\in\overline{C}|\,\mathcal{W}(i)=\lambda\right]}\\
&=\frac{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\mathcal{W}(i+1)=\Lambda\cap\left(\bigcap_{j=i+2}^{n}\mathcal{W}(j)\in\overline{C}\right)\right]}{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\left(\bigcap_{j=i+1}^{n}\mathcal{W}(j)\in\overline{C}\right)\right]}\\
&=\frac{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\mathcal{W}(i+1)=\Lambda\cap\left(\bigcap_{j=i+2}^{n}\Lambda+\sum_{k=i+2}^{j}\mathcal{X}(k)\in\overline{C}\right)\right]}{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\left(\bigcap_{j=i+1}^{n}\lambda+\sum_{k=i+1}^{j}\mathcal{X}(k)\in\overline{C}\right)\right]}.
\end{aligned}$$

Using independance of the random variables $\mathcal{X}$, one has then

$$\begin{aligned}
&\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]\\
&=\frac{\mathbb{P}\left[\mathcal{W}(i)=\lambda\cap\mathcal{W}(i+1)=\Lambda\right]}{\mathbb{P}\left[\mathcal{W}(i)=\lambda\right]}\frac{\mathbb{P}\left[\bigcap_{j=i+2}^{n}\Lambda+\sum_{k=i+2}^{j}\mathcal{X}(k)\in\overline{C}\right]}{\mathbb{P}\left[\bigcap_{j=i+1}^{n}\lambda+\sum_{k=i+1}^{j}\mathcal{X}(k)\in\overline{C}\right]},
\end{aligned}$$

and using the fact that they are identically distributed, one finally gets

$$\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1)=\Lambda\,|\,\mathcal{W}(i)=\lambda\right]=p(\lambda,\Lambda)\frac{\mathbb{P}\left[\Lambda+\mathcal{S}(\leq n-i-1)\in\overline{C}\right]}{\mathbb{P}\left[\lambda+\mathcal{S}(\leq n-i)\in\overline{C}\right]}.$$

$\square$

As desired, we finally obtain

Theorem 2.9 *If the drift is zero (equivalently : if $\theta = 1$ or $p_{+1} = p_{-1} = 1/2$), then*

$$\mathbb{P}_{\mathcal{W}(\leq n)\in\overline{C}}\left[\mathcal{W}(i+1) = \Lambda \,|\, \mathcal{W}(i) = \lambda\right] \xrightarrow[n\to\infty]{} \begin{cases} \frac{1}{2}\frac{\lambda+2}{\lambda+1} & \text{if } \Lambda = \lambda+1 \\ \frac{1}{2}\frac{\lambda}{\lambda+1} & \text{if } \Lambda = \lambda-1 \\ 0 & \text{otherwise} \end{cases} \qquad \lambda, \Lambda \in P^+,\ i \geq 0.$$

In other words, the sequence $\left(p_n^+(\,.\,,\,.\,,0)\right)_{n\geq 0}$ pointwise converges on $P^+ \times P^+$ and its limit $p^+(1)$ is given by

$$p^+(1)(\lambda,\Lambda) = p(1)(\lambda,\Lambda)\,\frac{\Lambda+1}{\lambda+1} \qquad \lambda,\Lambda \in P^+.$$

Thus, we have the pointwise convergence on $P^+ \times P^+$

$$p^+(\theta) \xrightarrow[\substack{\theta\to 1 \\ \theta\in]0,1[}]{} p^+(1)\,.$$

Proof. Use Proposition 2.8 and Theorem 2.6. $\square$

Law of the random walk with positive drift conditioned to stay nonnegative until instant n	$\xrightarrow[n\to\infty]{}$	Law of the random walk with positive drift conditioned to always stay nonnegative
$\downarrow \mathbf{d} = 0$		$\downarrow \mathbf{d} = 0$
Law of the random walk with zero drift conditioned to stay nonnegative until instant n	$\xrightarrow[n\to\infty]{}$	Law of the random walk with zero drift conditioned to always stay nonnegative

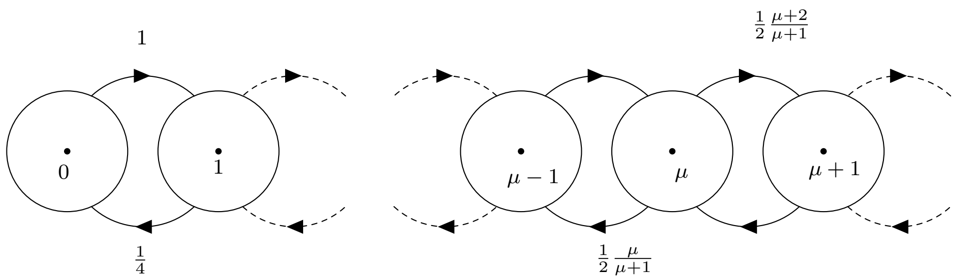

Figure 2: Graph of the Markov chain $\mathcal{W}^+$ in the zero drift case

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